Banach空间的凸性模与正规结构

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摘 要:令 *X*是 Banach空间,根据凸性 *U*-模与凸性 W^* -模和弱正交系数之间的关系来证明 Banach空间 *X*具有正规结构.由此加强了一些已知的结论.

关键词:凸性 *U*-模; 凸性 *W*^{*}-模; 弱正交系数; 一致正规结构; 不动点 **中图分类号**: 0177. 2 **文献标识码**: A **文章编号**: 1007- 2683 (2009) 02- 0026- 04

Modulus of Convexity and Normal Structure in Banach Space

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Abstract: Let X be a Banach space We present some sufficient conditions for which a Banach space X has normal structure by using the modulus of U - convexity, modulus of W^* - convexity and the coefficient of weak or thogonality.

Key words: modulus of U-convexity; uniform normal structure; fixed point

1 Introduction and Preliminaries

We assume that X and X^{*} stand for a Banach space and its dual space, respectively. By S_{X} and B_{X} we denote the unit sphere and the unit ball of a Banach space X, respectively. Let C be a nonempty bounded closed convex subset of a Banach space X.

Definition 1: A mapping T: C is said to be nonexpansive provided the inequality

$$// Tx - Ty // // x - y //$$

holds for every $x, y \in C$.

Definition 2: A Banach space X is said to have the fixed point property if every nonexpansive T: C = Chas a fixed point, where C is a nonempty bounded closed convex of a Banach space X.

Definition 3: A Banach space X is called to be u-

niform ly non - square if there exists >0 such that $\frac{1}{x + y} \frac{1}{2} = 1$ -

or

whenever x, $y = S_x$.

Definition 4: A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point, there exists a point x_0 H such that

 $\sup\{ || x_0 - y| |: y = H \} < \sup\{ || x - y| |: x, y = H \}$

Definition 5: A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure

In reflexive spaces, both notions coincide

Definition 6: A Banach space X is said to have uniform normal structure if there exists 0 < c < 1 such

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that for any closed bounded convex subset K of X that contains more than one point, there exists x_0 K such that

 $\sup\{/|x_0 - y|/: y \in K\} < c \sup\{/|x - y|/: x, y \in K\}$ It was proved by W. A. Kirk that every reflexive Banach space with normal structure has the fixed point property^{[8].}

The WORTH property was introduced by B. $\sin s^{[14]}$ as follows: a B anach space X has the WORTH property if

 $\lim_{n} | x_n + x - x_n - x | = 0$

for all $x \in X$ and all weak null sequences (x_n) .

Gao^[1-2] introduced the modulus of U - convexity, and modulus of W^* - convexity of a Banach space X, respectively, as follows:

$$U_{x}() = \inf\{1 - \frac{1}{2} \quad x + y \quad : x, y \quad S_{x},$$

$$f(x - y) \quad \text{for some } f \quad \nabla_{x}\}$$

$$W_{x}^{*}() := \inf\{\frac{1}{2}f(x - y) : x, y \quad S_{x},$$

$$x - y \quad \text{for some } f \quad \nabla_{x}\}$$

Here $\nabla_x := \{f \mid S_x : f(x) = x\}$. S Saejung^[10-11] studied the above modulus extensively, and got some useful results as follows:

1) If U_X () >0 or W^* () >0 for some (0,2), then X is uniform ly non - square.

2) If
$$U_x$$
 () $>\frac{1}{2}$ max { 0, -1 } for some

(0, 2), then X have uniform normal structure. Further if $U_X() > \max\{0, -1\}$ for some (0, 2), then X and X^* have uniform normal structure.

3) If W_x^* () > $\frac{1}{2} \max\{0, -1\}$ for some

(0, 2), then X and X have uniform normal structure.

In a recent paper [4], Gao introduced the following quadratic parameter, which was defined as:

$$E(X) = \sup\{ x + y^{2} + x - y^{2} : x, y = S_{X} \}$$

The constant is also significant tool in the geometric theory of Banach spaces Furthermore, Gao obtained the values of E(X) for some classical Banach spaces In term of the constant, he got some sufficient conditions for a Banach space X to have uniform normal structure, which played an important role in fixed point theory. **Definition 7**: Let *F* be filter on *N*. A sequence $\{x_n\}$ in *X* converges to *x* with respect to *F*, denoted by $\lim_F x_i = x$ if for each neighborhood *U* of *x*, $\{i \ N : x_i \ U\} = F$.

Definition 8: An ultrafilter is called trivial if it is of the form A:A N, i_0 A for some fixed i_0 N, otherwise, it is called nontrivial

Let l(X) denote the subspace of the product space $_{n N} X$ equipped with the norm (x_n) : = $\sup_{n N} x_n < \ldots$ Let μ be an ultrafilter on and let Nand let

 $N_u = \{ (x_n) \ l \ (X) : \lim] x_n = 0 \}$

The ultrapower of X, denoted by X, is the quotient space l / N_u equipped with the quotient norm. Write $(x_n)_u$ to denote the elements of the ultrapower Note that if μ is nontrivial, then X can be embedded into X isometrically.

2 Result and Proof

In this paper, we introduce a geometric constant as follows:

$$(t, X) = \sup\{ > 0: \lim_{n} \inf x_{x} + tx$$
$$\lim_{n} \inf x_{n} - tx \quad i = (0 < t - 1)$$

Where the supermum is taken over all the weakly sequence x_n in X and all elements x of X and $\frac{t}{3}$

(t, X) t the constant (x) is the case of $t = 1^{[15]}$. Then we will show that a Banach space X has uniform normall structure whenever U_X $(1 + (t, X)) > \frac{1 - (t, X)}{2}$ or W_X^* $(1 + (t, X)) > \frac{1 - (t, X)}{2}$. These

results improve the S Saejung 's and Gao 's results

Lemma $\mathbf{1}^{[5]}$ Let X be a Banach space without weak normal structure, then there exists a weakly null sequence $\{x_n\}_{n=1} \subseteq S_X$ such that

 $\lim_{x \to \infty} x_n - x = 1 \text{ for all } x = 0 \{x_n\}_{n=1}$

Theorem 1 Let X be a Banach space, if there exits 0 < t 1, such that $U_X (1 + (t, X)) > \frac{1 - (t, X)}{2}$, then X have uniform normal structure

Proof: By the hypothesis there exists 0 < t = 1such that $U_X (1 + (t, X)) > \frac{1 - (t, X)}{2}$, then it suffices to prove that X has weak normal structure whenev-

er
$$U_X (1 + (t, X)) > \frac{1 - (t, X)}{2}$$
. In fact, since $\frac{t}{3}$
 (t, X) t , we have $U_X (\cdot) > \frac{1 - (t, X)}{2} > 0$,

for some (0, 2), This implies that X is super - reflexive and then U_X () = U_{\Re} ()^[10]. Now suppose that X fails to have weak normal structure. Then, by the Lemma 1, there exists a weakly null sequence $\{x_n\}_{n=1}$ in S_X such that

$$\lim_{n} x_{n} - x = 1$$

for all $x = \cos\{x_n\}_{n=1}$.

Take $\{f_n\} \subset S_X$ such that $f_n = \nabla_{x_n}$ for all n = N. By the reflexivity of X^* , without loss of generality, we may assume that $f_n \xrightarrow{w^*} f$ for some $f = B_X \cdot (\text{where} \xrightarrow{w^*} \text{denote weak star convergence})$. We now choose a subsequence of $\{x_n\}_{n=1}$, denote again by $\{x_n\}_{n=1}$, such that

$$\lim_{n} x_{n+1} - x_n = 1,$$

$$/ (f_{n+1} - f) (x_n) / < \frac{1}{n}, \quad f_n(x_{n+1}) < \frac{1}{n}$$

for all n N. It follows that

 $\lim_{n \to 1} f_{n+1}(x_n) = \lim_{n \to 1} (f_{n+1} - f)(x_n) + f(x_n) = 0$ Put

$$\overline{\mathbf{W}} = (x_{n+1} - x_n)_u,$$

 $\overline{\mathbf{W}} = \begin{bmatrix} (t, X) (x_{n+1} + x_n) \end{bmatrix}_u$, and $f = (-f_n)_u$ by the definition of (t, x) and above Lemma, then

$$f = f(\mathbf{\overline{W}} = \mathbf{\overline{W}} =$$

1

and

较 = $[(t, X)(x_{n+1} + x_n)]_u$ $x_{n+1} - x_n = 1$ Furthermore, we have

$$f (\mathbf{\bar{w}} - \mathbf{\bar{w}}) = \lim_{u} (-f_n) ((1 - (t, X)) x_{n+1} - (1 + (t, X)) x_n) = 1 + (t, X)$$
$$\mathbf{\bar{w}} + \mathbf{\bar{w}} = \lim_{u} ((1 + (t, X)) x_{n+1} - (1 - (t, X)) x_n)$$
$$\lim_{u} (f_{n+1}) ((1 + (t, X)) x_{n+1} - (1 - (t, X)) x_n) = 1 + (t, X).$$

From the definiton of U_X (), we have

 $U_X (1 + (t, X)) = U_{\mathfrak{B}} (1 + (t, X)) - \frac{1 - (t, X)}{2}$

which is a contradiction Therefore U_X (1 + (t, X))

 $> \frac{1 - (t X)}{2}$ implies that X have uniform normal

structure.

Remark: the inequality $U_x (1 + (X)) > \frac{1 - (X)}{2}$ is the case of t = 1 in this paper^[13].

The modulus of convexity of X is the function x (): [0, 2] = [0, 1], defined by:

$$x() = \inf\{1 - \frac{x+y}{2} : x, y \in S_x, \\ x - y = \} = \inf\{1 - \frac{x+y}{2} : x = 1, \\ y = 1, x - y = \}.$$

The function x () strictly increasing on [0, (x), 2]. Here $0, (x) = \sup\{i: x(i) = 0\}$ is the charateristic of convexity of X. Also X is uniformly nonsquare provided 0, (x) < 2 On various geometrical properties and the geometric condition sufficient for normal structure in term of the modulus of convexity have been widely studied in paper [3], [5], [12], [16]. It is easy to prove that $U_x(i) = x(i)$, therefore we have the following corollary 2 which easy strengthens theorem 6 of Gao in paper [6].

Corollary 1 Let X be a Banach space, if there exits 0 < t 1, such that $_{X}(1 + (t, X)) > \frac{1 - (t, X)}{2}$, then X have uniform normal structure.

Remark: In fact corollary 1 is equivalent to Theorem 2 of paper [6], when t = 1.

It is well known that $_{0}(x) = 2_{x} \cdot (0)$, here $_{x}(0) = \lim_{t \to 0} \frac{x(t)}{t}$, $_{x}(t)$ is the modulus of smoothness be defined as

$$x(t) = \sup\{\frac{x+ty + x-ty}{2} - 1; x, y \in S_x\}.$$

Therefor we have the following corollary.

Corollary 2 Let X be a Banach space, if there exits 0 < t = 1, such that $_{X}(2 = (t, X)) > \frac{1 - (t, X)}{2}$, then X and X^{*} have uniform normal structure.

Proof: since
$$\frac{t}{3}$$
 (t, X)) t, from 2 (t, X)

t + (t, X) (0 < t = 1) and the monotone of x (), we have X have uniform normal structure from the corollary 1, we know that $(t, X) = (t, X^*)$ in a reflexive Banach space. So the inequality $x^* (0) < (t, X)$ or equivalently $_0 (x) < 2 (t, X)$ imply X^* have uniform normall structure. Form the definition of $_{0}(x)$, obviously the condition $_{x}(2 (t, X)) > \frac{1 - (t, X)}{2}$, imply $_{0}(x) < 2 (t, X)$. So X^{\star} have uniform normal structure.

Theorem 2 Let X be a Banach space, if there exits 0 < t = 1, such that $W_x^{\star}(1 + (t, X)) > \frac{1 - (t, X)}{2}$, then X have uniform normal structure.

Proof: It suffices to prove that X has weak normal structure whenever W_X^* $(1 + (t, X)) > \frac{1 - (t, X)}{2}$.

In fact, since $\frac{t}{3}$ (t, X)) t, we have W_X^* (2) >

 $\frac{1 - (t X)}{2}$ 0 for some (0, 2). This implies that X is super - reflexive, then W_X^* () = $W_{\mathfrak{H}}^*$ ()^[11]. Repeating the arguments in the proof of Theorem 1, and

$$\overline{\mathbf{W}} = (x_n - x_{n+1})_u,$$

 $\overline{\mathbf{w}} = [(t, X) (x_{n+1} + x_n)]_u$, and $f = (f_n)_u$ then

and

 $f(\mathbf{k}) = \mathbf{k} = 1$

1.

Furthermore, we have

$$\overline{\mathbf{w}} \cdot \overline{\mathbf{w}} = \lim_{u} ((1 + (t, X)) x_{n+1} - (1 - (t, X)) x_n) \lim_{u} (f_{n+1}) ((1 + (t, X)) x_{n+1} - (1 - (t, X)) x_n) = 1 + (t, X)$$

$$\frac{1}{2} \tilde{f} (\overline{\mathbf{w}} \cdot \overline{\mathbf{w}} = \frac{1}{2} \lim_{u} (f_n) ((1 - (t, X)) x_n - (1 + (t, X)) x_{n+1}) = \frac{1 - (t, X)}{2}.$$

But this implies W_X^* $(1 + (t, X)) = W_{\mathfrak{B}}^*$ $(1 + (t, X)) = \frac{1 - (t, X)}{2}$, which is a contradiction

Therefore W_X^* $(1 + (t, X)) > \frac{1 - (t, X)}{2}$, imply that

X have uniform normal structure

Remark: the inequality W_X^* $(1 + (X)) > \frac{1 - (X)}{2}$ is the case of $t = 1^{[16]}$.

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